AN EXACT SOLUTION OF THE NAVIER-STOKES EQUATIONS IN SPHERICAL COORDINATES

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1. INTRODUCTION

In this note, exact solutions of the steady incompressible Navier-Stokes equations in spherical coordinates have been obtained. The basic equations for fluid motions are the Navier-Stokes equation. These equations are non linear and only a limited number of exact solutions have been obtained. The existing exact solutions have been published in a wide variety of journals. A comprehensive recent review of exact solutions of Navier-Stokes equations is given by Wang [1]. The present paper deals with the exact solution in spherical coordinates [2].

2. Basic equations

The basic equations in the spherical coordinate system is governed by the continuity and momentum equations in the absence of body forces:

$$\nabla\cdot\mathbf{V}=\mathbf{0}, \qquad (1)$$

$$\rho\frac{D\mathbf{V}}{Dt}=-\nabla p+\mu\nabla^2\mathbf{V}, \qquad (2)$$

where

$$\mathbf{V}=\mathbf{V}\left(v_r(r,\theta,\phi),v_\theta(r,\theta,\phi),v_\phi(r,\theta,\phi)\right),$$

is the velocity vector, ρ is the constant density of fluid, $\frac{D}{Dt}=\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla$ is the material time derivative, ∇ is the nabla operator, p the pressure, μ is the dynamic viscosity of the fluid and ∇^2 is the Laplacian operator.

3. Formulation and Exact Solution of the Problem

Let r,θ,ϕ be coordinates with velocity components v_r, v_θ, v_ϕ, where ϕ denotes the coordinates parallel to the stagnation point. The v_r, v_θ, v_ϕ components of momentum equations are:

r-component of momentum equation:

$$\begin{aligned}
&v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{v_\theta^2+v_\phi^2}{r}=\\
&=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\Big[\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}+\\
&+\frac{2}{r}\frac{\partial v_r}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\\
&-\frac{2}{r^2}v_r-\frac{2\cot\theta}{r^2}v_\theta\Big],
\end{aligned} \qquad (3)$$

θ - *component* of momentum equation:

$$\begin{aligned}
&v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}+\frac{v_r v_\theta}{r}-\frac{\cot\theta}{r}v_\theta^2=\\
&=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+\nu\Big[\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2}+\\
&+\frac{2}{r}\frac{\partial v_\theta}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\\
&-\frac{1}{r^2\sin^2\theta}v_\theta\Big],
\end{aligned} \qquad (4)$$

ϕ - *component* of momentum equation:

$$\begin{aligned}
&v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial\theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_r v_\phi}{r}+\frac{\cot\theta\, v_\theta v_\phi}{r}=\\
&=-\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial\phi}+\nu\Big[\frac{\partial^2 v_\phi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2}\\
&+\frac{2}{r}\frac{\partial v_\phi}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial v_\phi}{\partial\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\phi}\\
&-\frac{1}{r^2\sin^2\theta}v_\phi\Big],
\end{aligned} \qquad (5)$$

where $\nu=\mu/\rho$ is the kinematic viscosity of the fluid. If the flow is independent of ϕ, the r and θ components of momentum Eqs. (2) can be solved for v_r and v_θ subject to continuity Eq. (1).

Let $\phi(r,\theta)$ be the potential function, where

$$v_r=-\phi_r,\quad v_\theta=-\frac{1}{r}\phi_\theta, \qquad (6)$$

where the suffixes denote differentiation and ϕ by definition satisfies $\nabla^2\phi=0$. Equations (3) and (4), in special cases, may be considered to be given by a potential function $¤\left(r,¯\right)$. Such type of flow satisfies the equations (3) and (4) if the pressure is given by

$$p = p_{o} - \frac{°}{2}\left[¤_{r}^{2} + \frac{1}{r^{2}} ¤_{¯}^{2} \right], \qquad (7)$$

where p_o is constant of integration.

In view of all the above discussion and assumptions, equation (5) reduces to

$$v_r \frac{\partial v_¤}{\partial r} + \frac{v_¯}{r}\frac{\partial v_¤}{\partial ¯} = ³ \left[\frac{\partial^2 v_¤}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_¤}{\partial ¯^2} + \frac{2}{r}\frac{\partial v_¤}{\partial r} + \right. \qquad (8)$$

$$\left. + \frac{\cot ¯}{r^2}\frac{\partial v_¤}{\partial ¯} \right]$$

Since equation (8) is a nonlinear partial differential equation. An exact solution [2] to equation (8) is:

$$v_¤ = A + Be^{-\frac{¤}{¬}}. \qquad (9)$$

where *A* and *B* are constants of integration.

3. Examples

(i) Flow along a corner

We consider as domain of the flow the region

$$D = \left\{ r \geq 0, ¯ \in \left[0, ® / 2\right], ¤ \in \left[0, ®\right] \right\},$$

If we seek a velocity $v_¤$ with the properties

$$v_¤(r,0) = v_¤\left(r, \frac{®}{2}\right) = 0,$$

$$v_¤(r,¯) \to V_0 \text{ as } r \to \infty,$$

then, a suitable form of the velocity $v_¤$ is

$$v_¤(r,¯) = V_0 - V_0 e^{-\frac{1}{¬}¤(r,¯)},$$

where the potential $¤(r,¯)$ satisfies conditions

$$¤(r,0) = ¤\left(r, \frac{®}{2}\right) = 0, \; ¤(r,¯) \to \infty \text{ as } r \to 0.$$

As a particular case we note

$$¤(r,¯) = kr\sin(2¯), \; k > 0,$$

with velocity field

$$v_r = -¤_r = -k\sin(2¯), \; v_¯ = -\frac{1}{r}¤_¯ = -2k\cos(2¯),$$

$$v_¤ = V_0 - V_0 e^{-\frac{k}{¬} r \sin(2¯)}.$$

(ii) Flow outside a sphere

Suppose we consider the region

$$D = \left\{ r \geq r_0, ¯ \in \left[0, 2®\right], ¤ \in \left[0, ®\right] \right\},$$

and seek a velocity with $v_¤(r_0,¯) = 0$ and $v_¤(r,¯) \to V_0$ as $r \to \infty$. A suitable form of $v_¤$ is

$v_¤(r,¯) = V_0 - V_0 e^{-\frac{1}{¬}¤(r,¯)}$ with $v_¤(r_0,¯) = 0,$ and $¤(r,¯) \to \infty$ as $r \to \infty$. As a particular case we note

$$¤(r,¯) = k(r - r_0)(2 + \sin ¯), \; k > 0,$$

$$v_r = -k(2 + \sin ¯), \; v_¯ = -\frac{k(r - r_0)}{r}\cos ¯,$$

$$v_¤ = V_0 - V_0 e^{-\frac{k}{¬}(r - r_0)(2 + \sin ¯)}.$$

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